Assignment 3

1. If p is an odd degree polynomial with real coefficients, then show, without using fundamental theorem of algebra, that it has a real root.

2. Let $p(x) = \sum_{k=1}^{n} a_k x^k$ and $q(x) = \sum_{k=0}^{m} b_k x^k$. Suppose $p(x) = q(x)$ for all $x \in [a, b]$. Show, without using fundamental theorem of algebra, that $n = m$ and $a_k = b_k$.

3. Let $D = \{x \mid 0 < x \leqslant 1\}$ and $f(x) = \frac{1}{x}$ defined on D. For every $a \in D$, let $L_a = \{x \in D \mid |f(x) - f(a)| < \frac{1}{3}\}$. Show that $\{L_a \mid a \in D\}$ covers D but there is no finite subcover.

4. Given the interval $[a, b]$, define $\alpha = \sup\{c \mid a \leqslant c < b \text{ and } [a, c] \text{ is compact}\}$. Show, without using Heine-Borel Theorem, that $\alpha = b$.
Hint: Use the definition of compactness.

5. If $\{K_n\}$ is a sequence of non-empty compact subsets of a metric space such that $K_n \supset K_{n+1}$ for all $n \in \mathbb{N}$, then show that $\cap_1^\infty K_n \neq \emptyset$.